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the partial series $\Sigma_r a_{i_r} x_{i_r}$ diverges while $\Sigma_r b_{i_r} x_{i_r}$ converges. Also the other $x$'s may be chosen so small as to make the corresponding values of $bx$ less than the terms of a given convergent series; so that we have made one series diverge while the other converges.

Also solved by S. BEATTY.

**483 (Algebra). Proposed by C. R. DUNCAN, Amherst College.**

Prove or disprove the following theorem: An infinite series $A_1 + A_2 + A_3 + \cdots + A_n + \cdots$ is convergent or divergent according as

$$\lim_{n=\infty} \frac{A_n}{1 - \dfrac{A_n}{A_{n-1}}} = 0 \quad \text{or} \quad \neq 0.$$

### SOLUTION BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

The test when applied to the series

$$1 + \tfrac{1}{1} + 2 + \tfrac{1}{2} + 3 + \tfrac{1}{3} + \cdots + n + \frac{1}{n} + \cdots$$

indicates convergence, whereas the series clearly diverges. If this example be considered artificial consider the series

$$\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \cdots + \frac{1}{n(\log n)^p} + \cdots.$$

Cauchy's integral test shows that this series diverges when $p \leqq 1$ and converges when $p > 1$.

$$\frac{A_n}{1 - \dfrac{A_n}{A_{n-1}}} = \frac{1}{\dfrac{1}{A_n} - \dfrac{1}{A_{n-1}}} = \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p}.$$

Now

$$\frac{1}{(\log n)^p} < \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p} < \frac{1}{[\log (n-1)]^p}.$$

Hence, the limit sought by the test is zero for all positive values of $p$, whereas it should be zero only for values greater than unity.

The suggested test for convergence is therefore no good.

If the limit exists and is greater than zero, then

$$\frac{1}{\dfrac{1}{A_n} - \dfrac{1}{A_{n-1}}} > \gamma \quad \text{or} \quad \frac{1}{A_n} - \frac{1}{A_{n-1}} < \frac{1}{\gamma}.$$

Hence

$$\frac{1}{A_{n+p}} - \frac{1}{A_{n-1}} < \frac{p+1}{\gamma},$$

$$A_{n+p} > \frac{1}{\dfrac{1}{A_{n-1}} + \dfrac{p+1}{\gamma}} > \frac{\gamma}{\dfrac{p+1+\gamma}{A_{n-1}}}.$$

(We have assumed a series of positive terms.) The series, therefore, diverges by comparison with the harmonic series. Hence, the test for divergence when applied to series of positive terms is valid.

Also solved by HORACE OLSON.

**514 (Geometry). Proposed by VICENTE MILLS, Manila, P. I.**

Given an equilateral triangle, the length of the sides being unknown, and a point within, the distances from which to the vertices are given, required the length of a side of the triangle and the angles subtended at the given point by the sides of the triangle.

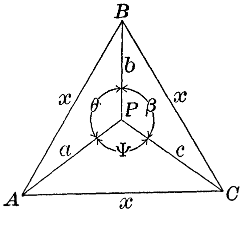
## I. Solution by William W. Johnson, Cleveland, Ohio.

From the figure, we have

$$\theta + \beta + \psi = 360^\circ, \qquad \theta = 360^\circ - (\beta + \psi)$$

and

$$\cos\theta = \cos(\beta + \psi) = \cos\beta\cos\psi - \sin\beta\sin\psi, \qquad \text{or} \qquad \cos\theta - \cos\beta\cos\psi = -\sin\beta\sin\psi.$$



Squaring both sides, we get

$$\begin{aligned}\cos^2\theta - 2\cos\theta\cos\beta\cos\psi + \cos^2\beta\cos^2\psi &= \sin^2\beta\sin^2\psi \\ &= (1 - \cos^2\beta)(1 - \cos^2\psi) \\ &= 1 - \cos^2\beta - \cos^2\psi + \cos^2\beta\cos^2\psi.\end{aligned}$$

Transposing and uniting, we have

$$\cos^2\theta + \cos^2\beta + \cos^2\psi - 2\cos\theta\cos\beta\cos\psi = 1. \qquad (1)$$

From the cosine formulas in trigonometry, we get

$$\cos\theta = \frac{a^2 + b^2 - x^2}{2ab} \ (A), \qquad \text{sos}\ \beta = \frac{b^2 + c^2 - x^2}{2bc} \ (B), \qquad \cos\psi = \frac{c^2 + a^2 - x^2}{2ca} \ (C).$$

Substituting these values in (1), clearing of fractions, and arranging terms, we get

$$x^6 - (a^2 + b^2 + c^2)x^4 + (a^4 + b^4 + c^4 - a^2c^2 - b^2a^2 - c^2b^2)x^2 = 0. \qquad (2)$$

Factoring, we find one root is $x^2 = 0$. Removing the factor $x^2$, we have

$$x^4 - (a^2 + b^2 + c^2)x^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0. \qquad (3)$$

This is a quadratic equation in $x^2$. Solving equation (3) for $x$, we find

$$x = \pm\sqrt{\frac{a^2 + b^2 + c^2}{2} \pm \tfrac{1}{2}\sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}}. \qquad (4)$$

The angles $\theta$, $\beta$, and $\psi$ may now be found by inserting the values of $x$ found in (4). The negative signs before the radicals apply when the point $P$ lies outside the triangle.

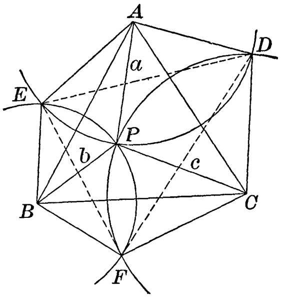
Having found $x$, the following formulas, adapted for logarithmic computation, may be used to replace $(A)$, $(B)$, and $(C)$.

$$\cos\tfrac{1}{2}\theta = \sqrt{\frac{p(p-x)}{ab}} \quad \text{in which} \quad p = \tfrac{1}{2}(a + b + x),$$

$$\cos\tfrac{1}{2}\beta = \sqrt{\frac{q(q-x)}{bc}} \quad \text{in which} \quad q = \tfrac{1}{2}(b + c + x),$$

$$\cos\tfrac{1}{2}\psi = \sqrt{\frac{r(r-x)}{ac}} \quad \text{in which} \quad r = \tfrac{1}{2}(a + c + x).$$

## II. Solution by Daniel Kreth, Wellman, Ia.



Let $ABC$ be the given triangle, and $P$ the given point. Put $AP = a$, $BP = b$, and $CP = c$. Let $x$ = a side of the triangle Then

$$\text{the area} = \frac{x^2}{4}\sqrt{3}. \qquad (1)$$

Construct the triangle $AEB$ = triangle $APB$, triangle $BFC$ = triangle $BPC$, and triangle $CDA$ = triangle $CPA$. Draw $DE$, $EF$, and $FD$.

Since $\angle DAE = 2\angle BAC = 120^\circ$, $DE$ is equal to a side of an equilateral triangle inscribed in a circle whose radius is $a$.

Hence, $DE = a\sqrt{3}$. In the same manner we find $EF = b\sqrt{3}$ and $FD = c\sqrt{3}$.

$$\text{Area of triangle } DAE = \frac{a^2}{4}\sqrt{3}; \tag{2}$$

$$\text{area of triangle } EBF = \frac{b^2}{4}\sqrt{3}; \tag{3}$$

$$\text{area of triangle } FCD = \frac{c^2}{4}\sqrt{3}; \tag{4}$$

and

$$\text{area of triangle } DEF = \tfrac{3}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}. \tag{5}$$

$$\begin{aligned}(2) + (3) + (4) + (5) &= \text{the area of the polygon } AEBFCD \\ &= 2 \times \text{area of triangle } ABC.\end{aligned}$$

Equating the two expressions for the area of the triangle, we have

$$\frac{x^2}{4}\sqrt{3} = \tfrac{1}{8}(a^2+b^2+c^2)\sqrt{3} + \tfrac{3}{8}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)};$$

$$\therefore\quad x = \sqrt{\tfrac{1}{2}\{(a^2+b^2+c^2) + \sqrt{3}\,[(a+b+c)(b+c-a)(a+c-b)(a+b-c)]\}}.$$

In the triangles we have all the sides given, to find the angles.

Also solved by NATHAN ALTSHILLER, NELLE L. INGELS and the PROPOSER.

**515 (Geometry). Proposed by C. F. GUMMER, Kingston, Ontario.**

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

### I. SOLUTION BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

**Construction.** Let $ABCD$ be the original square. Take $E$, $F$, $G$, $H$ as the middle points of its four sides. Lay off $BK = DJ$ equal to the distance $EF = GH$. Draw $EK$ and $JG$, which are obviously parallel. Remove the triangles $EBK$ and $JDG$, and place them in the positions $GCL$ and $KMN$, respectively. In any convenient position draw the line $OP$ perpendicular to $JG$ and $EK$. Remove the piece $AEPOJ$, and fit it into the position $MNRQL$. The square has now been transformed into the rectangle $OPRQ$, which is three times as long as wide, and may be divided by the lines $ST$ and $UV$ into three equal squares.